§ 3 Basics of Number Theory

Number Theory : Study of numbers (usually means integers)

Definition 3.1

Let $a, b \in \mathbb{Z}$, we say a divides b (denoted by $a \mid b$) if $b = ac$ for some $c \in \mathbb{Z}$.

In this case, a is said to be a divisor of b.

Example 3.1

$2 \mid 6$, $3 \mid 6$, $-3 \mid 6$, $3 \mid -6$, but $4 \nmid 6$

$n \mid 0$ for all integers n (A little bit odd to have $0 \mid 0$)

Definition 3.2

An integer $n > 1$ is said to be a prime if the only positive divisors of n are 1 and n, otherwise n is called a composite.

Remark: The number 1 is neither prime nor composite.

Example 3.2

First few primes : $2, 3, 5, 7, 11, 13, 17, 19, \cdots$

First few composites : $4, 6, 8, 9, 10, 12, 14, 15, \cdots$

Definition 3.3

Let $a, b \in \mathbb{Z}$. The greatest common divisor (gcd) of a and b is defined by

$$\gcd(a,b) = \begin{cases} \max\{d \in \mathbb{Z} : d \mid a \text{ and } d \mid b\} & \text{if not both } a, b \text{ are } 0 \\ 0 & \text{if } a = b = 0 \end{cases}$$

Remark : $\gcd(a, 0) = \max\{d \in \mathbb{Z} : d \mid a\} = |a|$

Example 3.3

Divisors of 18 : $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

Divisors of -12 : $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

$\gcd(18, -12) = 6$

Question: How to find $\gcd(a,b)$ if both a and b are large?

Theorem 3.1 (Division Algorithm)

Let $a, b \in \mathbb{Z}$ with $b \neq 0$ Then there exists unique $q, r \in \mathbb{Z}$ such that $0 \le r < |b|$ and $a = bq + r$.

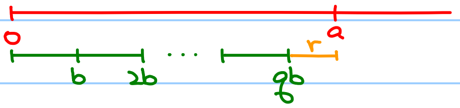


Diagram for the case of $a > b > 0$.

Lemma 3.1

$\gcd(a,b) = \gcd(b,r)$.

proof:

If $d = \gcd(a,b)$, then $d|a$ and $d|b$.

Therefore, $d \mid a - bq = r$.

$d|b$ and $d|r$ (d is a common divisor of b and r) $\Rightarrow d \le \gcd(b,r)$

If $d' = \gcd(b,r)$, then $d'|b$ and $d'|r$.

Therefore, $d' \mid bq + r = a$

$d'|a$ and $d'|b$ (d is a common divisor of a and b) $\Rightarrow d' \le \gcd(b,r)$

$\therefore \gcd(a,b) = \gcd(b,r)$.

Example 3.4 (Euclidean Algorithm)

Find $\gcd(240, 168)$

$240 = 1 \times 168 + 72 \qquad \gcd(240,168) = \gcd(168,72)$

$168 = 2 \times 72 + 24 \qquad \gcd(168,72) = \gcd(72,24)$

$72 = 3 \times 24 \qquad \gcd(72,24) = 24$

$\therefore \gcd(240,168) = 24$

Exercise 3.1

Find $\gcd(817, 1247)$.

Ans. 43

Theorem 3.2

Let $a, b \in \mathbb{Z}$. There exists $s, t \in \mathbb{Z}$ such that $as + bt = \gcd(a,b)$.

Example 3.5 (Extended Euclidean Algorithm)

$284 = 4 \times 68 + 12$

$68 = 5 \times 12 + 8$

$12 = 1 \times 8 + 4$

$8 = 2 \times 4$

$$
\begin{aligned}
\gcd(284, 68) = 4 &= 12 - 1 \times 8 \\
&= 12 - 1 \times (68 - 5 \times 12) \\
&= 6 \times 12 - 1 \times 68 \\
&= 6 \times (284 - 4 \times 68) - 1 \times 68 \\
&= 6 \times 284 - 25 \times 68
\end{aligned}
$$

Example 3.6

bucket with unknown volume | glass 105 mL | cup 180 mL | water tap

Question: What should we do so that at the end we have 15 mL of water in the bucket?

By extended Euclidean Algorithm, $\gcd(180, 105) = 15 = 3 \times 180 + (-5) \times 105$

Question: Will it end up with 10 mL of water in the bucket?

Exercise 3.2

Let $a, b, c \in \mathbb{Z}$. Prove that

There exists $s, t \in \mathbb{Z}$ such that $as + bt = c$ if and only if $\gcd(a,b) \mid c$.

(Therefore, $\{as + bt : s, t \in \mathbb{Z}\}$ = set of all multiples of $\gcd(a,b)$.)

Definition 3.4

Let $a, b \in \mathbb{Z}$. a and b are said to be relatively prime if $\gcd(a,b) = 1$.

(i.e. a and b has no common factor other than ± 1.)

Lemma 3.2

Let $n, a, b \in \mathbb{Z}$ such that $n \mid a$ and $n \mid b$, then $n \mid \gcd(a,b)$.

(i.e. every common divisor of a and b is also a divisor of $\gcd(a,b)$.)

proof:

$n \mid a$ and $n \mid b \Rightarrow a = np$ and $b = nq$ for some $p, q \in \mathbb{Z}$.

There exist $s, t \in \mathbb{Z}$ such that $\gcd(a,b) = as + bt = n(ps + qt)$ where $ps + qt \in \mathbb{Z}$.

$\therefore n \mid \gcd(a,b)$

Proposition 3.1

Let $a, b \in \mathbb{Z}$ and let p be a prime. If $p | ab$, then $p | a$ or $p | b$.

proof:

Suppose that $p | ab$.

If $p | a$, it's done!

If $p \nmid a$, since p is a prime, we have $\gcd(a, p) = 1$

Then, there exist $s, t \in \mathbb{Z}$ such that $1 = as + pt$

$b = abs + ptb$

$b = pqs + ptb$ $\quad p | ab \Rightarrow ab = pq$ for some $q \in \mathbb{Z}$.

$b = p(qs + tb)$

$\therefore p | b$

Theorem 3.3 (Prime Factorization)

Every positive integer greater than 1 can be expressed as a product of primes in a unique way

proof:

Let S be the set of all positive integers greater than 1 which cannot be expressed as a product of primes.

Suppose the contrary Then S is a nonempty set of $\mathbb{N}$.

By well ordering principle, S has a least element m. Firstly, m cannot be a prime, so $m = ab$ for some positive integers a, b with $a, b < m$.

Therefore, $a, b \notin S$, i.e. a and b can be expressed as a product of primes, but then $m = ab$ which can be expressed as a product of primes. (Contradiction)

$\therefore$ Every positive integer greater than 1 can be expressed as a product of primes.

Suppose that n is a positive integer greater than 1 and $n = p_1 p_2 \cdots p_r = q_1 q_2 \cdots q_s$ where p_i's and q_i's are primes.

By proposition 3.1, $p_1 | q_1 q_2 \cdots q_s \Rightarrow p_1 | q_i$ for some i

but q_i itself is a prime, so $q_i = p_1$

By swapping the index, we let $q_1 = p_1$ and we have $p_2 \cdots p_r = q_2 \cdots q_s$

Repeating the above, we have $r = s$ and $p_i = q_i$ for $i = 1, 2, \cdots, r$

$\therefore n$ can be expressed as a product of primes a unique way.

Primes : "Elements" of numbers !

Exercise 3.3

Let $a, b, c \in \mathbb{Z}$. Show that if $c \mid ab$ and $\gcd(a,c) = 1$, then $c \mid b$

Some Results / Questions of Number Theory :

1) Question : How many primes ?

Theorem 3.4

There are infinitely many primes.

2) Question : Given a positive integer n, how many primes $\leq n$ are there ?

Let $\pi(n) = |\{p \in \mathbb{Z}^+ : p \leq n \text{ is a prime}\}|$.

Theorem 3.5

$\lim_{n\to\infty} \frac{\pi(n)}{\left(\frac{n}{\log n}\right)} = 1$ (Some would like to state it as $\lim_{n\to\infty} \frac{\pi(n)}{\left(\frac{n}{(\log n)-1}\right)} = 1$.)

Think : $\pi(1000) = 168 \approx \frac{1000}{(\log 1000) - 1} \approx 169.27$

3) Twin primes : both p and $p+2$ are primes, e.g. $(3,5), (5,7), (11,13), (17,19)$

Question : Are there infinitely many pairs of twin primes ?

Not yet known (Twin prime conjecture)

4) Note. $3^2+4^2=5^2$, $5^2+12^2=13^2$, $7^2+24^2=25^2$

Question : Given an integer $n > 2$, are there positive integers a, b, c such that

$$a^n + b^n = c^n \ ?$$

Answer : No ! (Fermat Last Theorem)

The Ring of Integers Modulo n

Definition 3.5

Let n be a positive integers.

If $a, b \in \mathbb{Z}$ such that $n \mid b-a$, then we say a is congruent to b modulo n,

and it is denoted by $a \equiv b \pmod{n}$

Remark. " | " defines an equivalence relation $\sim$ on $\mathbb{Z}$ ($a \sim b$ if $n \mid b-a$)

Proposition 3.2

If $a \equiv a' \pmod{n}$, $b \equiv b' \pmod{n}$, then $a+b \equiv a'+b' \pmod{n}$ and $ab \equiv a'b' \pmod{n}$

(Define $\sim$ on $\mathbb{Z}$ so that $a \sim b$ if $n \mid b-a$. The above proposition means

If $a \sim a'$ and $b \sim b'$, then $a+b \sim a'+b'$ and $ab \sim a'b'$.

Addition and multiplication on $\mathbb{Z}$ induce addition and multiplication on $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/\sim$.)

Example 3.7

$23 \equiv 2 \pmod{7}$, $34 \equiv 6 \pmod{7}$

$23 + 34 \equiv 2+6 \equiv 8 \equiv 1 \pmod{7}$ (Compare to $23+34 \equiv 57 \equiv 1 \pmod{7}$)

$23 \times 34 \equiv 2 \times 6 \equiv 12 \equiv 5 \pmod{7}$ (Compare to $23 \times 34 \equiv 782 \equiv 7 \times 111 + 5 \equiv 5 \pmod{7}$)

Another interpretation: Consider $[23] = [2]$, $[34] = [6] \in \mathbb{Z}/7\mathbb{Z}$,

$$[23+34] = [23]+[34] = [2]+[6] = [2+6] = [8] = [1]$$

$$[23 \times 34] = [23] \times [34] = [2] \times [6] = [2 \times 6] = [12] = [5]$$

As a set $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], [2], \cdots, [n-1]\}$ which contains $n-1$ elements, but we would also like to know the algebraic structures on $\mathbb{Z}/n\mathbb{Z}$ (such as addition and multiplication).

It turns out that

(i) $\mathbb{Z}/n\mathbb{Z}$ is a ring

(ii) $\mathbb{Z}/p\mathbb{Z}$ is a field if p is a prime. (Discuss later!)

Proposition 3.2 (Cancellation)

If $\gcd(c,n) = 1$ and $ac \equiv bc \pmod{n}$, then $a \equiv b \pmod{n}$

proof:

$n \mid ac-bc = (a-b)c$ and $\gcd(c,n) = 1 \Rightarrow n \mid a-b$ i.e. $a \equiv b \pmod{n}$ (see exercise 3.3)

Example 3.8

$4 \times 1 \equiv 4 \times 4 \pmod{6}$ but $1 \not\equiv 4 \pmod{6}$ since $\gcd(4,6) = 2 \neq 1$.

$ax \equiv b \pmod{n}$

Question: How to solve $ax \equiv b \pmod{n}$?

Proposition 3.3

$ax \equiv b \pmod{n}$ is solvable if and only if $\gcd(a,n) \mid b$

proof:

The equation can be solved $\Leftrightarrow$ There exist $x, q \in \mathbb{Z}$ such that $ax + nq = b$

$\Leftrightarrow \gcd(a,n) \mid b$ (see exercise 3.2)

In particular, if p is a prime and $p \nmid a$, then $ax \equiv b \pmod{p}$ is solvable.

Also, if x_1 and x_2 are solutions of $ax \equiv b \pmod{p}$,

$a(x_1 - x_2) \equiv b - b \equiv 0 \pmod{p}$ and $\gcd(a,p) = 1$

then we have $p \mid x_1 - x_2$ (or $x_1 \equiv x_2 \pmod{p}$)

$\therefore$ All solutions are congruent modulo p.

Example 3.9

Solve $4x \equiv 3 \pmod{9}$

Note that $\gcd(4,9) = 1$, the above equation is solvable.

$9 - 4 \times 2 = 1$ —(*) (By extended Euclidean algorithm)

$9 \times 3 + 4 \times (-2) = 3$

$4 \times (-6) \equiv 3 \pmod{9}$

$\therefore$ -6 is one of the solution of $4x \equiv 3 \pmod{9}$

(*) shows that $4 \times (-2) \equiv 1 \pmod{9}$ (or $4 \times 7 \equiv 1 \pmod{9}$ if you like)

-2 acts as an "inverse" of 4

In general, $4x \equiv b \pmod{9}$

$(-2)(4x) \equiv -2b \pmod{9}$

$x \equiv -8x \equiv -2b \pmod{9}$ (Note $-8 \equiv 1 \pmod{9}$)

Another intepretation : Find $[x] \in \mathbb{Z}/9\mathbb{Z}$ such that $[4][x] = [3]$

Note : $[-2][4] = [1]$ (or $[7][4] = [1]$)

We have $[4][x] = [3]$

$[-2][4][x] = [-2][3]$

$[1][x] = [-6]$

$[x] = [-6]$ (or $[3]$) ($\therefore$ solution to $4x \equiv 3 \pmod 9$ are those $x \in [3]$)

$a^m \equiv 1 \pmod n$

Question : Given $a, n \in \mathbb{Z}$ and $a \neq 0$, does it exist $m \in \mathbb{Z}^+$ such that $a^m \equiv 1 \pmod n$?

Firstly, $a^m \equiv 1 \pmod n$ for some $m \in \mathbb{Z}^+$

$\Rightarrow a \cdot a^{m-1} + nq = 1$ for some $q \in \mathbb{Z}$ (Convention. $a^0 = 1$)

$\Rightarrow \gcd(a,n) = 1$

However, if $\gcd(a,n) = 1$, does it exist $m \in \mathbb{Z}^+$ such that $a^m \equiv 1 \pmod n$?

Think : There are only n elements of $\mathbb{Z}/n\mathbb{Z}$, but $[a], [a^2], [a^3], \cdots \in \mathbb{Z}/n\mathbb{Z}$,

so there exists $i, j \in \mathbb{Z}^+$ with $i < j$ such that $[a^j] = [a^i]$ i.e $a^j \equiv a^i \pmod n$

Since $\gcd(a,n) = 1$, we can cancel a's and so $a^{j-i} \equiv 1 \pmod n$

Definition 3.6

Let $a, n \in \mathbb{Z}$ such that $\gcd(a,n) = 1$.

The order of a modulo n is the least $m \in \mathbb{Z}^+$ such that $a^m \equiv 1 \pmod n$

Example 3.10

Table of a^m modulo 6

$a \backslash m$	1	2	3	4	5
0	0	0	0	0	0
1	1	1	1	1	1
2	2	4	$8 \equiv 2$	$16 \equiv 4$	$32 \equiv 2$
3	3	$9 \equiv 3$	$27 \equiv 3$	$81 \equiv 3$	$243 \equiv 3$
4	4	$16 \equiv 4$	$64 \equiv 4$	$256 \equiv 4$	$1024 \equiv 4$
5	5	$25 \equiv 1$	$125 \equiv 5$	$625 \equiv 1$	$3125 \equiv 5$

$\gcd(0,6), \gcd(2,6), \gcd(3,6), \gcd(4,6) \neq 1$

$\gcd(1,6), \gcd(5,6) = 1$

Order of $1 = 1$

Order of $5 = 2$

Definition 3.7

The Euler φ function is defined by $\varphi(n) = |\{a \in \mathbb{Z}^+ : a \leq n \text{ and } \gcd(a,n)=1\}|$ for $n \in \mathbb{Z}^+$.

$\varphi(1) = |\{1\}| = 1$ $\qquad$ $\varphi(2) = |\{1\}| = 1$ $\qquad$ $\varphi(3) = |\{1,2\}| = 2$

$\varphi(4) = |\{1,3\}| = 2$ $\qquad$ $\varphi(5) = |\{1,2,3,4\}| = 4$ $\qquad$ $\varphi(6) = |\{1,5\}| = 2$

In particular, if p is a prime, $\varphi(p) = p-1$;

if p and q are primes, $\varphi(pq) = (p-1)(q-1)$;

if p and q are relatively prime, $\varphi(pq) = \varphi(p)\varphi(q)$.

Theorem 3.6 (Euler's Theorem)

If $\gcd(a,n) = 1$, then $a^{\varphi(n)} \equiv 1 \pmod{n}$.

Let $(\mathbb{Z}/n\mathbb{Z})^\times = \{[a] \in \mathbb{Z}_n : \gcd(a,n) = 1\}$ (and so $|(\mathbb{Z}/n\mathbb{Z})^\times| = \varphi(n)$).

Definition 3.8

A primitive root is an element of $(\mathbb{Z}/n\mathbb{Z})^\times$ of order $\varphi(n)$.

Example 3.11

Table of a^m modulo 15 (with $\gcd(a,15)=1$).

a \ m	1	2	3	4
1	1			
2	2	4	8	1
4	4	1		
7	7	4	13	1
8	8	4	2	1
11	11	1		
13	13	4	7	1
14	14	1		

Note: $\varphi(15) = 8$

Order of $1 = 1$

Order of $4, 11, 14 = 2$

Order of $2, 7, 8, 13 = 4$ (No primitive root)

Table of a^m modulo 5 (with $\gcd(a,5)=1$)

a \ m	1	2	3	4
1	1			
2	2	4	3	1
3	3	4	2	1
4	4	1		

Note: $\varphi(5) = 4$

Order of $1 = 1$

Order of $4 = 2$

Order of $2, 3 = 4$

(2 and 3 are primitive roots)

Idea of proof of Euler's Theorem:

1) Prove that if $[a], [b] \in (\mathbb{Z}/n\mathbb{Z})^\times$, then $[a][b] = [ab] \in (\mathbb{Z}/n\mathbb{Z})^\times$.

2) Let $[a] \in (\mathbb{Z}/n\mathbb{Z})^\times$ and let $f: (\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow (\mathbb{Z}/n\mathbb{Z})^\times$ defined by $f([x]) = [a][x] = [ax]$.

Prove that f is bijective.

3) $\prod_{[x]\in(\mathbb{Z}/n\mathbb{Z})^\times} [x] = \prod_{[x]\in(\mathbb{Z}/n\mathbb{Z})^\times} [ax] = [a^{\varphi(n)}] \prod_{[x]\in(\mathbb{Z}/n\mathbb{Z})^\times} [x]$

$[a^{\varphi(n)}] = [1]$ i.e. $a^{\varphi(n)} \equiv 1 \pmod{n}$ (Note: $[x] \in (\mathbb{Z}/n\mathbb{Z})^\times$ and by definition $\gcd(x,n) = 1$, so it can be cancelled.)

Modular Exponentiation

How to compute 5^{5210} modulo 21 ?

First of all, $\gcd(5,21) = 1$, so we have $5^{\varphi(21)} \equiv 1 \pmod{21}$.

Also $\varphi(21) \overset{(*)}{=} \varphi(3\times 7) = \varphi(3)\varphi(7) = 2\times 6 = 12$

$\therefore 5^{5210} \equiv 5^{434\times 12+2} \equiv (5^{12})^{434} \times 5^2 \equiv 25 \equiv 4 \pmod{21}$

Remark: (*) involves factorization of an integer which may not be done easily! There exists no algorithm to factorize an integer which can be done in polynomial time.

Exercise 3.4

Compute 7^{1234} modulo 72

Hint: $\gcd(7,72) = 1$, $\varphi(72) = \varphi(8)\varphi(9)$. Ans: 25

How to compute 26^{13} modulo 196 ?

Note: $\gcd(26, 196) = 2 > 1$, so we may not use Euler's Theorem.

$26^1 \equiv 26 \pmod{196}$

$26^2 \equiv 26\times 26 \equiv 676 \equiv 88 \pmod{196}$

$26^4 \equiv 26^2 \times 26^2 \equiv 88\times 88 \equiv 7744 \equiv 100 \pmod{196}$

$26^8 \equiv 26^4\times 26^4 \equiv 100\times 100 \equiv 10000 \equiv 4 \pmod{196}$

$\therefore 26^{13} \equiv 26^{1+4+8} \equiv 26\times 100\times 4 \equiv 52\times 4 \equiv 208 \equiv 12 \pmod{196}$

Remark: Every number in red is less than 195^2.

Chinese Remainder Theorem

有物不知其數，　　　Let $x \in \mathbb{Z}$.

三三數之剩二，　　　$x \equiv 2 \pmod 3$

五五數之剩三，　　　$x \equiv 3 \pmod 5$

七七數之剩二。　　　$x \equiv 2 \pmod 7$

問物幾何？　　　$x = ?$

孫子算經

Theorem 3.7 (Chinese Remainder Theorem)

Let $a_1, a_2, \cdots, a_k \in \mathbb{Z}$ and $n_1, n_2, \cdots, n_k \in \mathbb{Z}^+$ such that $\gcd(n_i, n_j) = 1$ for all $i \neq j$.

There exists $x \in \mathbb{Z}$ such that

$$x \equiv a_1 \pmod{n_1}$$

$$x \equiv a_2 \pmod{n_2}$$

$$\vdots$$

$$x \equiv a_k \pmod{n_k}$$

proof:

Let $N = n_1 n_2 \cdots n_k$ and $N_i = \frac{N}{n_i} = n_1 \cdots n_{i-1} n_{i+1} \cdots n_k$

Note: $\gcd(n_i, n_j) = 1$ for all $i \neq j$

$\Rightarrow \gcd(n_i, N_i) = 1 \Rightarrow$ there exist $m_i, M_i \in \mathbb{Z}$ such that $n_i m_i + N_i M_i = 1 \Rightarrow N_i M_i \equiv 1 \pmod{n_i}$

Also $M_i N_i \equiv 0 \pmod{n_j}$ for $j \neq i$.

Then $x = \sum_{i=1}^{k} a_i M_i N_i$ is a solution.

Furthermore, if $x_1, x_2 \in \mathbb{Z}$ are solutions, then

$$x_1 - x_2 \equiv 0 \pmod{n_i} \quad \text{for } 1 \leq i \leq k.$$

$\therefore x_1 - x_2 \equiv 0 \pmod N$

$a_1=2$, $a_2=3$, $a_3=2$, $n_1=3$, $n_2=5$, $n_3=7$

$N=3\times5\times7=105$, $N_1=35$, $N_2=21$, $N_3=15$

$35\times\underset{\substack{\uparrow\\ M_1}}{2}+3\times(\underset{\substack{\uparrow\\ m_1}}{-23})=1$ $\qquad 21\times\underset{\substack{\uparrow\\ M_2}}{1}+5\times(\underset{\substack{\uparrow\\ m_2}}{-4})=1$ $\qquad 15\times\underset{\substack{\uparrow\\ M_3}}{1}+7\times(\underset{\substack{\uparrow\\ m_3}}{-2})=1$

三人同行七十希，

五樹梅花廿一支，

七子團圓正半月，

除百零五便得知。

$x\equiv 2\times\underset{\substack{\uparrow\\ N_1M_1}}{70}+3\times\underset{\substack{\uparrow\\ N_2M_2}}{21}+2\times\underset{\substack{\uparrow\\ N_3M_3}}{15}\equiv 233\equiv 23 \pmod{105}$

Example 3.12

Find $x\in\mathbb{Z}$ such that $x\equiv3\pmod 8$ and $x\equiv5\pmod 9$.

$n_1=8$, $n_2=9$ and so $\gcd(n_1,n_2)=\gcd(8,9)=1$

$a_1=3$, $a_2=5$

$N=n_1n_2=8\times9=72$

$N_1=\dfrac{N}{n_1}=9=n_2$ $\qquad N_2=\dfrac{N}{n_2}=8=n_1$

$9\times1+8\times(-1)=1$

$N_1M_1+n_1m_1=1$

$n_2m_2+N_2M_2=1$

Let $x\equiv \underbrace{3\times9\times1+5\times8\times(-1)}_{a_1N_1M_1+a_2N_2M_2}\equiv-13\equiv59 \pmod{72}$

Exercise 3.5

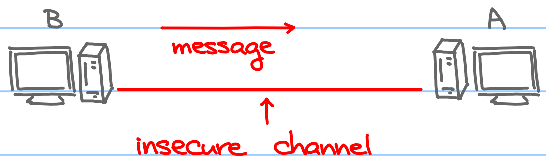
a) Find $x\in\mathbb{Z}$ such that $x\equiv3\pmod 7$ and $x\equiv13\pmod{15}$.

Ans: $x\equiv73 \pmod{105}$

b) Find $x\in\mathbb{Z}$ such that $x\equiv2\pmod 7$, $x\equiv3\pmod 8$ and $x\equiv6\pmod 9$

Ans: $x\equiv51 \pmod{504}$

RSA crytosystem



Question: How to use an insecure channel to transmit data in a secure way?

Exercise: Try to factorize 8137.

Ans: $8137 = 79 \times 103$ (Difficult?)

Idea of RSA crypstosystem: Difficult to factorize a product of two large primes!

RSA algorithm:

Key generation by A:

1) Choose two large primes p, q and compute $n = pq$.
2) Compute $\varphi(n) = \varphi(pq) = (p-1)(q-1)$ and keep private.
3) Choose $1 < e < \varphi(n)$ such that $\gcd(e, \varphi(n)) = 1$ (For example, choose a prime e and $e \nmid \varphi(n)$)
4) Find d such that $ed \equiv 1 \pmod{\varphi(n)}$ (This equation is solvable as $\gcd(e, \varphi(n)) = 1$)

   Keep d private.

Operation:

1) The pair of numbers (n, e) (called public key) is released by A.
2) Suppose $0 \le m < n$ is the message to be sent from B to A,

   B sends $c \equiv m^e \pmod{n}$ to A instead. (c is called ciphertext).
3) A computes c^d modulo n, and the result is m, i.e. $c^d \equiv m^{ed} \equiv m \pmod{n}$

Lemma 3.3

$c^d \equiv m^{ed} \equiv m \pmod{n}$

proof:

By Chinese remainder theorem, m is a solution of $(*)\begin{cases} x \equiv m \pmod{p} \\ x \equiv m \pmod{q} \end{cases}$

therefore, for any solution x of $(*)$, we have $x \equiv m \pmod{n}$.

Thus, what we need to show are $m^{ed} \equiv m \pmod{p}$ and $m^{ed} \equiv m \pmod{q}$,

i.e. m^{ed} is also a solution of $(*)$, then $m^{ed} \equiv m \pmod{n}$.

Claim: $m^{ed} \equiv m \pmod{p}$

Recall: $ed \equiv 1 \pmod{\varphi(n)} \Rightarrow ed = 1 + k\varphi(n) = 1 + k(p-1)(q-1) = 1 + k\varphi(p)\varphi(q)$ for some $k \in \mathbb{Z}$.

1) If $\gcd(m,p) = 1$, then $m^{\varphi(p)} \equiv 1 \pmod{p}$ (Euler's theorem)

and so $m^{ed} \equiv m^{1+k\varphi(p)\varphi(q)} \equiv m \cdot (m^{\varphi(p)})^{k\varphi(q)} \equiv m \cdot 1^{k\varphi(p)} \equiv m \pmod{p}$

2) If $\gcd(m,p) \neq 1$, then $p \mid m$ and so $m^{ed} \equiv 0 \equiv m \pmod{p}$

Similarly, we can show that $m^{ed} \equiv m \pmod{q}$.

Example 3.13

Key generation by A:

1) Choose two primes $p = 11$, $q = 17$ and compute $n = pq = 187$

2) Compute $\varphi(n) = \varphi(pq) = (p-1)(q-1) = 10 \times 16 = 160$ and keep private.

3) Choose $1 < e < \varphi(n)$ such that $\gcd(e, \varphi(n)) = 1$ (For example, choose a prime $e = 19$)

4) Find d such that $ed \equiv 1 \pmod{\varphi(n)}$ i.e. $19d \equiv 1 \pmod{160}$

By extended Euclidean algorithm, $19 \times 59 + 160 \times (-7) = 1$, i.e. $\underset{e}{19} \times \underset{d}{59} \equiv 1 \pmod{\underset{\varphi(n)}{160}}$

Keep $d = 59$ private.

Operation:

1) Public key $(n, e) = (187, 19)$ is released by A.

2) Suppose $0 \leq m = 32 < 187$ is the message to be sent from B to A,

B sends the ciphertext $c \equiv m^e \equiv 32^{19} \equiv 43 \pmod{n = 187}$ to A instead.

3) A computes $m \equiv c^d \equiv 43^{59} \equiv 32 \pmod{n = 187}$

Exercise 4.5

Find c if we use $m = 53$ (Ans: $c = 93$), verify your answer by computing c^d modulo n.